Towards a Cognitive Semantics of Types

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Abstract. Types are a crucial concept in conceptual modelling, logic, and knowledge representation as they are an ubiquitous device to understand and formalise the classification of objects. We propose a logical treatment of types based on a cognitively inspired modelling that accounts for the amount of information that is actually available to a certain agent in the task of classification. We develop a predicative modal logic whose semantics is based on conceptual spaces that model the actual information that a cognitive agent has about objects, types, and the classification of an object under a certain type. In particular, we account for possible failures in the classification, for the lack of sufficient information, and for some aspects related to vagueness.

Keywords: Types, Conceptual Spaces, Sortals, Identity, Vagueness

1 Introduction

Conceptual Modelling is a discipline of fundamental importance for several areas in Computer Science, including Software Engineering, Enterprise Architecture, Domain Engineering, Database Design, Requirements Engineering and, in particular, for several subareas of Artificial Intelligence, most notably, Knowledge Representation and Ontology Engineering [7].

From a cognitive point of view, without types, we would not be able classify objects and, without classification, our mental life would be chaotic. As [21] puts it: *Categorization* [...] *is a means of simplifying the environment, of reducing the load on memory, and of helping us to store and retrieve information efficiently.* If we perceived each entity as unique, we would be overwhelmed by the sheer diversity of what we experience and unable to remember more than a minute fraction of what we encounter. Furthermore, if each individual entity needed a distinct name, our language would be staggeringly complex and communication virtually impossible. In contrast, if you know nothing about a novel object but you are told it is an instance of X, you can infer that the object has all or many properties that Xs have [20].

Frequently, monadic types used in conceptual models have as their instances *objects*, i.e., entities that persist in time, possibly undergoing qualitative changes, while keeping their identity throughout most of these changes. On the one hand, the importance of object types is well recognized in the aforementioned areas, as basically all conceptual modelling, ontology design, and knowledge representation approaches have as first-class citizens modelling primitives to represent them. On the other hand, in most of these approaches, the notion of object types taken is equivalent to the notion of a unary predicate in first-order logical languages. As a consequence, these approaches ignore a number of fundamental ontological and cognitive aspects related to the notion of object type. These include the following.

Not all types have the same ontological nature, hence, not all types classify objects in the same manner and with the same force. In particular, whilst all types provide a *principle of application* for deciding whether something fall under their classification, only types of a particular sort (termed *sortals*) provides also a *principle of persistence, individuation, counting and trans-world identity* for their instances [12, 16]. In particular, a specific type of sortal termed a *kind* is fundamental for capturing the *essential properties* of the objects they classify. Hence, kinds classify their instances *necessarily* (in the modal sense), i.e., in all possible situations. In fact, there is a large amount of empirical evidence in cognitive psychology supporting the claim that we cannot make any judgment of identity without the support of a kind [22, 16]. This, in turn, should have direct consequences to our formal understanding of types. For instance, given that there is no identity and no counting of individuals without a kind, then we should not quantify over individuals of our domains without the support of a kind. On a second aspect, from a cognitive point of view, our judgment of both the qualities and attributes of an object as well of which objects fall under a certain type are *vague*.

In this paper we make a contribution towards a new logical system designed to capture important ontological and cognitive aspects of types. Firstly, we propose a view of types that is grounded in the information that a cognitive agent has about the type, the object that may be apt to be classified under the type, and about the act of classification. In order to ground this view on a cognitively motivated account of types, we place our modelling within the theory of conceptual spaces proposed by Gärdenfors [6]. Briefly, Gärdenfors models the qualities of an objects by means of multidimensional space, a *conceptual space*, endowed with a distance that is intended to represent similarity between qualities. An example is the space of colours, composed by a three dimensions (brightness, chromaticness, hue) that are capable of explaining the similarity perceived by a cognitive agent among colours, by means of the methodology of multidimensional scaling [10].

Secondly, this system is compatible with philosophical view of objects and types, which focuses on the crucial distinctions between sorts of types, in particular, differentiating between sortals and ordinary predicates and between essential (kinds) and inessential sortals. In order to characterize these distinctions, a system must heavily rely on a modal treatment of objects and types. For this reason, it is compelling to understand how the modal view can be combined with the idea of the classification captured by conceptual spaces.

A combination of modal logic and conceptual spaces has been developed in [8] where the semantics of types may be given in terms of *concepts* in the sense of Gärdenfors, i.e., suitable subsets of a conceptual space. In this paper, we aim to extend the conceptual view of types by introducing a number of elements that better account for possible failures, partiality, and vagueness of the task of classification that cognitive agents might face. In order to achieve that, a number of significant points of departures from Gärdenfors' view has to be embraced.

On the one hand, we shall view the semantics of types as *rough sets* rather than mere sets. Rough sets are a generalisation of sets that allows for indeterminacy in the classification [17]. A rough set is intuitively composed by three regions: its positive part, containing the case of certain classification, its negative part, containing the cases of non-classification, and a boundary of undetermined classification. Rough sets are then a viable tool for representing partiality, vagueness, or uncertainty in the task of classification.

On the other hand, the view of objects in conceptual spaces is problematic both from a cognitive and a philosophical perspective [15]. Objects in the view of Gärdenfors are identified with points in a conceptual space. From a cognitive perspective, that amounts to viewing the information about a certain object as fully determined. For instance, an agent must know the precise shade of colour, the exact weight, the sharp length of any recognisable object. This is indeed cognitively unrealistic. From a philosophical perspective, by identifying objects as points in a conceptual space, we are identifying the reference of a proper name with its specified sensible qualities and this aspect poses serious problems to the modalisation of the view of objects. In particular, we lose the rigid designation of proper names, as it is hard to assume that the qualities of an object are stable across possible worlds, time, circumstances. Abandoning the rigid designation of proper names entail facing a serious amount of complications effectively highlighted by Kripke [11]. For that reason, we prefer to embrace rigid designation of proper names (i.e. the individual constants in our logical language) and abandon the identification of objects with points in a conceptual space.

To cope for this two aspects, the cognitive and the philosophical perspective, we shall assume that individual constants are interpreted in a separated set of objects that is distinguished by the conceptual space. The denotation of constants is then rigid, in any possible world the denotation of individual constants is the same. However, the amount of information about the object may vary. For this reason we shall also associate the individual constants to the amount of information that an agent has about the qualities of the objects. The information associated to the individual constants plays the role of the *intension* in a Fregean perspective: roughly, the intension captures the amount of information that is required in order to access or recognise the extension, or the denotation, of the constant [5]. In a similar way, we shall introduce the intension of a type as the amount of information that the agent has about the the type and we shall define the extension of a type as set of objects classified by the type. According to a Fregean perspective, as we shall see, the intension of a type will allow for determining its extension.

The contribution of this paper is the following. We present a three-valued logic to reason about the classification of objects under possibly rough types emerging form concepts in a conceptual space. For this reason, we extend the treatment of classification of [8] to account for partial and vague information. Moreover, we develop a modal logic on top of the three-valued logic in order to formally capture the significant definitions of kinds of types, therefore extending [15] and [14] to account for modalities. The remainder of this paper is organised as follows. The next section presents a simplified formulation of conceptual spaces. Section 3 presents a three-valued logic for rough sets that is adequate to represent classification under the assumption of vague or partial information. Section 4 discusses the proof-theory of the proposed system. Section 5 presents an applications of our framework to the classification of types. Section 6 concludes.

2 Conceptual spaces

We present our formal framework that relies on previous work on conceptual spaces. Our definition of conceptual spaces is inspired by the formalizations based on vector spaces provided in [1] and [19]. A *domain* Δ is given by a number of n quality dimensions $Q_1, \dots, Q_n$ endowed with a distance d_Δ that usually depends on the distances defined on its quality dimensions. Following [1], we assume that every domain contains the distinguished point $*$.[1] A conceptual space is defined by Gärdenfors as a set of domains $\{\Delta_1, \dots, \Delta_n\}$. We simplify the model by putting the following definition:

Definition 1. *A conceptual space is a subset of the cartesian product of* n *domains:* $\mathcal{C} \subseteq \Delta_1 \times \cdots \times \Delta_n$.

Our definition is weaker than the one proposed by [1], as we are taking any subset of the cartesian product as a conceptual space. This is motivated just as a simplifying move. Stronger definitions, that express, for instance, separability and integrality of the domains, can be retrieved by putting suitable constraints on $\mathcal{C}$.

A point of a conceptual space with n domains is an element $x \in \mathcal{C}$, that is, $x = \langle x_1, \dots, x_n \rangle$ is a vector of values in each domain, i.e., we do not explicitly consider the dimensions of the domains and the reduction of the distances d_{Δ_i} to the ones of the dimensions. These aspects are not relevant to the present treatment.

Gärdenfors represents *concepts* as regions in conceptual spaces. In addition, he assumes that *natural* concepts correspond to sets of convex regions (that represent natural properties) in a number of domains with a salience assignment. We leave salience for future work and, to provide an interpretation to disjunctions and negations of concepts, we do not concentrate on what Gärdenfors terms natural concepts, e.g., the union or the complement of convex regions, in general, is not convex. A *sharp concept* is then represented just as a subset $R \subseteq \mathcal{C}$. By contrast, we represent *rough concepts* by *rough sets* [17]. Following [2], a rough set of $\mathcal{C}$ is specified by means of a pair of sets $\langle A, B \rangle$ such that $A \subseteq B \subseteq \mathcal{C}$: A

[1] $*$ means here that a certain quality is not applicable to a certain object.

represents the interior of the rough set, B is its exterior, and $B \setminus A$ its boundary. The intuition behind a rough concept C is that one does not have a sharp definition of C, i.e., only the properties that belong to its interior are necessary for all the instances of C. The properties that belong to its boundary are, in general, satisfied only by some instances of C. Therefore, for the objects placed at the boundary of C, one can neither conclude that they are C-instances nor that they are not C-instances. This aspect relates to *prototype theory*, where one can consider a *graded membership* with two thresholds, one associated with A, the other with B. These thresholds can also be defined on the basis of the distances d_Δ defined on the relevant dimension. Finally, note that sharp concepts are just a special case of rough ones, i.e., they can simply be represented by particular rough sets with form $\langle A, A \rangle$.

3 A logic for types

We introduce a predicative language $\mathcal{L}$ by specifying the alphabet that contains a countable set of individual constants $\mathbf{C} = \{c_1, c_2, \dots\}$ and a countable set of variables $\mathbf{V} = \{x_1, x_2, \dots\}$. The set of unary predicates (i.e. types) is split into two sorts of predicates: *kind* $\mathbf{K} = \{K_1, K_2, \dots\}$ and *regular*, or common, predicates, $\mathbf{P} = \{P_1, P_2, \dots\}$ that have, as we shall see, distinct interpretations. We focus on unary predication in this paper, thus we do not introduce relations at this point, besides the identity $\{=\}$, which we shall discuss in a dedicated section. In particular, we also assume a set of (relative identity) relations $\{=_{K_1}, \dots, =_{K_n}\}$, where K_i are kinds. As we shall see, the relative identity relations apply only to objects of the same kind.

Moreover, we assume the following set of logical connectives $\{\neg, \wedge, \vee, \rightarrow\}$. For quantification, we assume as in [8] only a restricted form of quantification that depends on a kind, that is, we assume the following set of restricted quantifiers $\{\forall_{K_1}, \forall_{K_2}, \dots,\}$ where $K_i \in \mathbf{K}$. Moreover, we assume a modal operator on propositions $\{\Box\}$.

The set of formulas is defined by induction in the usual way. Assume that $c_i, c_j \in \mathbf{C}$, $Q \in \mathbf{S} \cup \mathbf{P}$, $\star \in \{\neg, \wedge, \vee, \rightarrow\}$, and $\phi(x)$ denotes the formula ϕ with x as the sole free variable[2].

$$\mathcal{L} ::= c_i =_{K_i} c_j \mid Q(c_i) \mid \phi \star \phi \mid \forall_{S_j} \phi(x) \mid \Box \phi$$

We introduce now the structure that we use to define the models of our language [15].

Definition 2. *A* conceptual *structure for $\mathcal{L}$ is a tuple $S = \langle \mathcal{C}, \mathcal{O}, \epsilon, \iota, \sigma \rangle$ where:*

- *$\mathcal{C}$ is a conceptual space;*
- *$\mathcal{O}$ is a non empty set of* objects*;*
- *ϵ is a function that maps individual constants into objects, $\epsilon : \mathbf{C} \rightarrow \mathcal{O}$;*

[2] Since we deal with restricted quantification, we define the predicative language without open formulas.

– *ι is a function that maps predicates into rough sets of $\mathcal{C}$, $\iota : \mathbf{P} \rightarrow \mathcal{P}(\mathcal{C}) \times \mathcal{P}(\mathcal{C})$;*
– *σ is a function that maps objects into regions of $\mathcal{C}$, $\sigma : \mathcal{O} \rightarrow \mathcal{P}(\mathcal{C})$.*

A *conceptual model* M is then obtained by adding a valuation function $||\cdot||_M$ that maps formulas to a suitable set of truth-values. $||\cdot||_M$ depends on ϵ, ι, σ but also on the choice of the set of truth-values that captures the possible classifications. To capture reasoning about rough sets, we assume three truth-values $\{t, u, f\}$, representing truth, false, and undetermined.

The modal structure of the logic is then captured by means of the following definitions.

Definition 3. *A* constant domain frame $\langle W, R, \mathcal{C}, O\rangle$ *is given by a set of possible worlds W, an accessibility relation R, a conceptual space $\mathcal{C}$ and a domain of interpretation of the individual constants, that is in our case, the set of objects O.*

An interpretation $\mathcal{I}_w$ specifies, for each $w \in W$, a conceptual model M for w. That is, a conceptual structure $S = \langle \mathcal{C}, \mathcal{O}, \epsilon, \iota_w, \sigma_w\rangle$ such that $\mathcal{C}$, O, and ϵ are fixed for every possible world w. What indeed can change through possible worlds is the intension of individual constants σ_w and of predicates ι_w and therefore, as we shall see, their extension when varying w.

Formally, the objects in $\mathcal{O}$ provide the denotation of the individual constants in $\mathbf{C}$. The function ϵ, called *extension*, associates individual constants to objects, thus it plays the role of the interpretation function in a standard first-order model. In Fregean terms, ϵ provides the denotation, or the reference, of the individual constant.

The σ function locates objects in $\mathcal{C}$, i.e., it characterises an object in terms of its properties, represented as regions of $\mathcal{C}$. In this sense, the view of the meaning of individual constants defined here is similar to their treatment in terms of individual concepts proposed in [8]. Different objects may then have the same associated properties, allowing us to deal with the problem of coincidence and separate it from identity. In addition, the partial or vague information about objects can be represented by locating them in regions (rather than points) of $\mathcal{C}$, in case the exact (fully determinate) properties of an object are not known. To capture the case of fully determinable objects, it is always possible to assign all the objects to singleton subsets of $\mathcal{C}$. Again, in Fregean terms, σ provides the intension of the individual constant, the amount of information about the object that is required in order to understand its denotation.

The (rough) concepts of $\mathcal{C}$ provide the semantics of both kinds and regular predicates of $\mathcal{L}$. The function ι_w, called *intension*, maps a predicate $P \in \mathbf{P}$ into a (rough) set of $\mathcal{C}$ that we indicate by $\langle \underline{\iota_w}(P), \overline{\iota_w}(P)\rangle$.[3] Given a rough set, we

[3] Note that we are using a concept of intension that differs from the standard view of modal logics, i.e. the Carnap view of intensions as functions form possible worlds to extensions of predicates. Here the intension of a type, in a Fregean perspective, is the relevant information associated to the type. This information may or may not change through possible worlds. This is an open question that requires further investigation. In this paper, we prefer not to commit to either assumption.

can define three regions (that depend on the interpretation of P in w): $POS^w_P = \underline{\iota_w}(P)$, $NEG^w_P = \mathcal{C} \setminus \bar{\iota_w}(P)$, and $BN^w_P = \bar{\iota_w}(P) \setminus \underline{\iota_w}(P)$.

The (rough) extension of a predicate P in the world w—the set of objects that (roughly) satisfy P in w—can be defined on the basis of how an object is positioned in $\mathcal{C}$ (via σ_w) with respect to the intension of P in w. This is captured by means of the following definition:

$$\epsilon_w(P) = \{o \in \mathcal{O} \mid \sigma_w(o) \subseteq \iota_w(P)\} \tag{1}$$

The extension of a predicate in w is given by the set of objects that are classified by $\iota_w(P)$. This view embraces a Fregean perspective on predicates and on predication: the intension of a predicate (its *Sinn*) is a mean to obtain, or compute, its extension, cf. [5]. In this way, we partially capture the intension of predicates, i.e., their meaning is not reduced to a mere set of objects, rather it is given by the amount of information that is required to perform the classification [18].

Since fully determinate objects are associated to points and sharp sets are a special case of rough sets, our modelling generalises the case of classification defined in [6] and [8].

3.1 Semantics of predication

We start by presenting the semantics of atomic sentences. We denote by $||\phi||_w$ the valuation of a formula ϕ in w, meaning that ϕ has value $||\phi||_w$ at w. Since objects are associated to regions in $\mathcal{C}$ and predicates are mapped to rough sets of $\mathcal{C}$, we have to decide how to view the amount of information about the object and the concept [15]. Among the viable readings, we choose the following definition.

$$||P(a)||_w = t \text{ iff } \sigma_w(\epsilon(a)) \subseteq POS^w_P \tag{2}$$

$$||P(a)||_w = u \text{ iff } \sigma_w(\epsilon(a)) \nsubseteq POS^w_P \text{ and } \sigma(\epsilon(a)) \nsubseteq NEG^w_P \tag{3}$$

$$||P(a)||_w = f \text{ iff } \sigma_w(\epsilon(a)) \subseteq NEG^w_P \tag{4}$$

True means that all the points in $\sigma(\epsilon(a))$ are certainly P, that is, they are in the positive part of P. The case of falsity means that all the points of $\sigma(\epsilon(a))$ are certainly not P, that is, they are in the negative part of P. The case of $||P(a)||_w = u$ does not entail that $\sigma(\epsilon(a))$ is fully included in BN^w_P, $\sigma(\epsilon(a))$ may spread across all the three regions POS^w_P, NEG^w_P, and BN^w_P, cf. Figure 3.1.

3.2 Kinds vs Regular types

We define kinds as predicates that are rigid and that provide an identity criterion. To capture the distinction between kinds and regular predicates, we restate the previous definition as follows. For $P \in \mathbf{P}$, $||Q(a)||_w$, is exactly as in (2), (3), and (4). For $K \in \mathbf{K}$, we fix the interpretation of K across possible worlds:

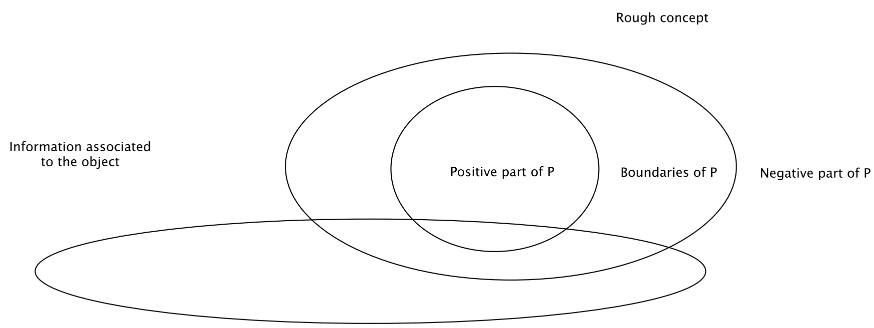

Fig. 1. Undertermined truth-value of classification

$$||K(a)||_w = t \text{ iff for all } w,\ \sigma_w(\epsilon(a)) \subseteq POS^w_K \quad (5)$$

$$||K(a)||_w = u \text{ iff for all } w,\ \sigma_w(\epsilon(a)) \nsubseteq POS^w_K \text{ and } \sigma(\epsilon(a)) \nsubseteq NEG_{K_w} \quad (6)$$

$$||K(a)||_w = f \text{ iff for all } w,\ \sigma_w(\epsilon(a)) \subseteq NEG^w_K \quad (7)$$

The definition of kinds amounts to assuming that their extension is fixed in any possible worlds. For instance, an object that is classified as (of the kind) person cannot cease to be a person. In principle, we could also fix the intension of each kind for every world, that would amount to assuming that the information associated to a kind is the same in any possible world.

A second crucial feature of kinds is that they provide identity criteria. That is, identity is only defined across individuals classified by the same kind (either directly classified under the same kind or classified under sortals that specialize the same kind). For this reason, we present now the treatment of the identity relation.

3.3 Identity

We can in principle add a standard identity relation to the first order language. The semantics is given by the following definition. For any two terms τ and τ', we have that:

- $||\tau = \tau'||_w = t$ iff $\varepsilon(\tau) = \varepsilon(\tau')$.
- $||\tau = \tau'||_w = f$ iff $\varepsilon(\tau) \neq \varepsilon(\tau')$.

As the definition shows, the truth-value of every identity statement only depends on ε. This view of identity is *ontological*, in the sense that it does not depend on any data about τ and τ'. Moreover, identity statements are determined, there is indeed no room for the third truth-value u.

This view entails, due to the constant domain assumption, that if an identity statement is true at a world, then it is necessary true. In this sense, that notion of identity captures an ontological view of identity.

In our language, instead, we assume that ontological identity is always relative to a kind. That is why we assumed the set of relative identity relations $\{=_{K_1}, \ldots, =_{K_n}\}$.

$$||\tau =_{K_1} \tau'||_w = t \text{ iff } \sigma(\varepsilon(\tau)) \subseteq POS^w_{K_l},\ \sigma(\varepsilon(\tau')) \subseteq POS^w_{K_l} \text{ and } \varepsilon(\tau) = \varepsilon(\tau'). \tag{8}$$

$$||\tau =_{K_l} \tau'||_w = f \text{ iff } \sigma(\varepsilon(\tau)) \subseteq POS^w_{K_l},\ \sigma(\varepsilon(\tau')) \subseteq POS^w_{K_l},\ \varepsilon(\tau) \neq \varepsilon(\tau'). \tag{9}$$

$$||\tau =_{K_1} \tau'||_w = u \text{ iff otherwise} \tag{10}$$

In case we also have a standard identity predicate, the relative identity relations can be defined as follows.

- $\tau =_{K_i} \tau' \leftrightarrow K_i(\tau) \wedge K_i(\tau') \wedge \tau = \tau'$

As we have seen, the standard identity is always determined. Relative identity is not, this is due to the possible uncertainty in classifying an object under a kind, e.g. the truth-value of $K_i(\tau)$ may be u. For this reason, the relative identity has a cognitive note, it depends on the actual information that are available about the relata and their classification under a kind.

3.4 Semantics of logical operators

The definition of logical connectives follows the following truth tables of Kleene logic plus the Lukasiewicz definition of implication, which is suited for treating a semantics based on rough sets [2].

$\neg$	t	u	f
	f	u	t

$\wedge$	t	u	f
t	t	u	f
u	u	u	f
f	f	f	f

$\vee$	t	u	f
t	t	t	t
u	t	u	u
f	t	u	f

$\rightarrow$	t	u	f
t	t	u	f
u	t	t	u
f	t	t	t

The semantic definition of restricted quantifiers is the following. Denote by $Q(x/c_j)$ the substitution of the variable x with constant c_j.

$$||\forall_{K_l} x\, Q(x)||_w = t \text{ iff for all } c_j \text{ such that } \sigma(c_j) \subseteq POS^w_{K_l},\ ||Q(x/c_j)||_w = t \tag{11}$$

$$||\forall_{K_l} x\, Q(x)||_w = f \text{ iff there is a } c_j \text{ such that } \sigma(c_j) \not\subseteq POS^w_{K_l},\ ||Q(x/c_j)||_w = f \tag{12}$$

$$||\forall_{K_l} x\, Q(x)||_w = u \text{ iff otherwise.} \tag{13}$$

The true clause of restricted quantification means that a universal statement is true if for any constant c_j whose interpretation satisfies the kind K_l (i.e. $\sigma(c_j) \subseteq POS_{K_l}$), $Q(x/c_j)$ is true at w. The false clause only required that there is a counterexample to $\forall_{K_l} x\, Q(x)$ among those c_j that fall under the kind K_l. In any other case, the value of $\forall_{K_l} x\, Q(x)$ is undetermined.

The semantics of the $\Box$ modality is then the following:

$$||\Box\phi||_w = t \text{ iff for all } w' \text{ such that } wRw',\ ||\phi||_w = t \tag{14}$$

$$||\Box\phi||_w = f \text{ iff there is a } w' \text{ such that } wRw',\ ||\phi||_w = f \tag{15}$$

$$||\Box\phi||_w = u \text{ iff otherwise.} \tag{16}$$

In a similar fashion as the case of universal quantification, we assume that a $\Box\phi$ is true if for every accessible world, ϕ is true there, $\Box\phi$ is false if there is an accessible world that falsifies ϕ, and it is undetermined in the other cases. Note that, in case we want to maintain a standard modal logic setting, kinds and restricted quantification can be defined by maintaining the following axioms. Denote by $A(x)$ a formula with x among its free variables.

$$K(x) \leftrightarrow \Box K(x) \tag{17}$$

$$\neg K(x) \leftrightarrow \Box\neg K(x) \tag{18}$$

$$\forall_K x A(x) \leftrightarrow \forall x (K(x) \to A(x)) \tag{19}$$

The first two axioms fix the extension of a kind in every possible world, while the third axiom defines the relative identity relations.

4 A Hilbert system

We can introduce a Hilbert system for capturing reasoning in our proposed system. The first-order Lukasiewicz three-valued logic is defined by means the the following list of axioms and two inference rules developed in [3] and [9]. For the modal part, since we are assuming a constant domain for the interpretation of the individual constants (via σ), we shall assume the Barcan formula and its converse [4]. Moreover, we assume the principle K, i.e. axiom 16, and the necessitation rule.

The concept of derivation in $\vdash$ is defined by induction as usual [3]. We leave the proof of soundness and completeness for a dedicated future work.

In this calculus, the distinction between kinds and regular predicates, the restricted quantifications, and the relative identity relations can be defined as we have seen in Definitions (17), (18), and (19).

5 Sortal, rigid, anti-rigid types

We defined a kind K as a predicate that is rigid and that provides an identity criterion, by enabling to define a relative identity relation $=_K$. For the sake

Axioms

1. $A \rightarrow (B \rightarrow A)$
2. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
3. $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)$
4. $((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)$
5. $(((A \rightarrow B) \rightarrow A) \rightarrow A) \rightarrow (B \rightarrow C) \rightarrow (B \rightarrow C)$
6. $A \wedge B \rightarrow A$
7. $A \wedge B \rightarrow B$
8. $(A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow B \wedge C))$
9. $A \rightarrow A \vee B$
10. $B \rightarrow A \vee B$
11. $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$
12. $(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$
13. $\forall x A(x) \rightarrow A(\tau)$
14. $\forall x(A \rightarrow B) \rightarrow (A \rightarrow \forall x B)$ (where x is not free in A).
15. $\forall x \Box A(x) \leftrightarrow \Box \forall x A(x)$
16. $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

Rules

- *Modus Ponens*: if $\vdash A$ and $\vdash A \rightarrow B$, then $\vdash B$.
- *Generalisation*: if $\vdash A$, then $\vdash \forall x A(x)$.
- *Necessitation*: if $\vdash A$, then $\vdash \Box A$

Table 1. Axioms and rule for modal Lukasiewicz logic

of example, we present how to define a few other important notions of types. Firstly, we define the concepts of rigidity, non-rigidity, and anti-rigidity.

- We say that a predicate P is *rigid* if $\Box \forall x(P(x) \rightarrow \Box P(x))$
- We say that a predicate P is *non-rigid* if $\neg \Box \forall x(P(x) \rightarrow \Box P(x))$
- We say that a predicate P is *anti-rigid* if $\forall x(\neg \Box P(x))$

We define a *sortal* type S as the conjunction (i.e. the specialisation) of a kind K by means of a predicate that may not be rigid.

A predicate S is a *sortal* iff $S(x) \leftrightarrow (K(x) \wedge P(x))$ where P is any predicate (20)

That is, a sortal is logically equivalent to the specialisation of a kind by means of any predicate. Even if K is rigid, since P may be any predicate, the sortal S may or may not be rigid (there may be worlds in which $P(x)$ is not true while $K(x)$ is). The sortal S inherits however the identity criterion provided by the kind K: two elements of S must be elements of K, therefore the identity criterion provided by $=_K$ applies to them. Notice that in case also P is a kind, we have that the sortal S is also rigid. Therefore, in this case, the sortal is also a kind.

In this case, the sortal is a kind that is logically equivalent to the conjunction of two other kinds.

Moreover, we define what the philosopher David Wiggins [13] terms a *phased sortal* as the specialisation of a kind by means of an anti-rigid predicate.

$$\text{A predicate } S \text{ is a } \textit{phased sortal} \text{ iff } S(x) \leftrightarrow (K(x) \wedge P(x)) \text{ where } P \text{ is anti-rigid} \tag{21}$$

For instance, the type *student* can be defined as the specialization of the kind *person* by the anti-rigid property of being enrolled in some course of study.

For completeness, we can also define types that are provided by the specialisation of a kind by means of a non-rigid predicate.

Further properties of kinds can be added by means of logical constraints. For instance, the assumptions that individuals are partitioned into kinds can be easily expressed in this language. We leave that for a dedicated future work.

6 Conclusion

We have introduced a logic for types that combines a cognitively inspired semantics and a modal treatment. We embraced a Fregean perspective in separating, for individual constants and types, their intension, which provides the actual amount of information available, and their extension. By using rough sets on a conceptual space, we modelled the possibly vague intension of types and, by associating regions in conceptual spaces to objects, we modelled the possibly vague or partial information about the objects. As we have seen, the logic that is adequate to model predication under this assumptions requires three truth values. We extended this treatment with modalities in order to capture the fundamental classifications of types, e.g. kinds, sortals, phased sortals.

Future work concerns two directions. On the one hand, we are interested in establishing the adequacy and the computational complexity of the proposed logical system to the reasoning about possibly uncertain classifications under types and in investigating its viable extensions. On the other hand, we intend to use this framework to provide an cognitively motivated exhaustive classification of the variety of types actually used in Conceptual Modelling and Knowledge Representation.

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