On the Representation of Quantities and their Parts in Conceptual Modeling

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Abstract. In a series of publications, we have employed ontological theories and principles used to evaluate and improve the quality of conceptual modeling grammars and models. In this article, we continue this work by providing an ontological interpretation and sound modeling guidelines for a traditionally neglected notion in the conceptual modeling literature, namely, the representation of types whose instances are quantities (amounts of matter, masses). Here, we analyze different alternatives for the adequate representation of quantities as well as their parts in conceptual models. Moreover, we advance a number of metamodeling constraints that can be incorporated in a UML 2.0 metamodel extension, thus, allowing for the suitable representation of these notions.

Keywords. Conceptual Modeling, Real-World Semantics, Representation of Quantities

Introduction

In recent years, there has been a growing interest in the application of *Foundational Ontologies*, i.e., formal ontological theories in the philosophical sense, for providing real-world semantics for conceptual modeling languages, and theoretically sound foundations and methodological guidelines for evaluating and improving the individual models produced using these languages.

In a series of publications, we have successfully applied ontological theories and principles to analyze a number of fundamental conceptual modeling constructs ranging from Roles, Types and Taxonomic Structures, Part-Whole Relations, Relationships, Attributes, Weak Entities and Datatypes, among others [1-4]. In this article we continue this work by addressing a representation problem which has been traditionally neglected not only in the literature of conceptual modeling but also in existing ontological analysis of conceptual modeling grammars. We focus here on the representation of *quantities* (amounts of matter, masses). Existing proposals for modeling quantities in conceptual modeling deal exclusively with the representation of a *quantitative measure* aspect of quantities (e.g., [5, p.36]). However, as we argue in this article, there are many modeling situations in which this approach does not suffice. For this reason, we need a modeling approach for explicitly dealing with the complementary *numerical identity* aspect.

The contributions of this article are two-fold. Firstly, we conduct an ontological analysis of the notion of quantities and explore alternative manners in which these entities could be adequately represented in conceptual models. Our criteria of adequacy here include fundamental requirements for conceptual modeling which includes that the types representing quantity universals in a conceptual model have only instances which are determinate, i.e., which obey determinate *principles of identity*. Moreover, we also consider a fundamental requirement that conceptual models including quantity types should be satisfiable by finite data populations. As a result, we advocate for a specific interpretation to the notion of quantities and a specific proposal for their representation.

Secondly, given the specific proposed ontological interpretation for quantities, we explore the nature of a parthood relation involving quantities and their subparts. We then explicitly define this relation both in terms of its traditional mereological meta-properties (e.g., transitivity, extensionality) but also in terms of two additional characteristics which are fundamental for conceptual modeling, namely, *shareability* and *essentiality*. As an additional result connected to this second contribution, we present a number of metamodeling constraint that can be used for the implementation of a UML[1] modeling profile for representing quantities and their subparts.

The remainder of this article is organized as follows. Section 1 briefly reviews a number of aspects related to part-whole theories which are germane to the purposes of this article. Section 2 revisits the notion of principle of identity and its connection to sortal universals. The section also briefly explores the connection between identity and the notion of *homeomerosity* for the case of quantities, as both notions are later shown to play an important role in evaluating representation alternatives. Section 3 analyzes three different alternative solutions for representing quantities in conceptual modeling. Section 4 elaborates on the representation of a specific type of parthood relation between quantities and their subparts. Finally, section 6 presents some final considerations.

1. A Review of Part-Whole Theories

1.1. Mereological Theories

In practically all philosophical theories of parts, the relation of proper parthood (symbolized as <) stands for a strict partial ordering, i.e., an asymmetric and transitive relation, from which irreflexivity follows:

$$\forall x\ \neg(x < x) \quad \textbf{(1)}$$
$$\forall x,y\ (x < y) \rightarrow \neg(y < x) \quad \textbf{(2)}$$
$$\forall x,y,z\ (x < y) \wedge (y < z) \rightarrow (x < z) \quad \textbf{(3)}$$

These axioms amount to what is referred in the literature by the name of *Ground Mereology (M)*, which is the core of any theory of parts. Taking reflexivity (and antisymmetry) as constitutive of the meaning of 'part' implies regarding identity as a limit case of parthood. In this spirit, an *improper part of* relation (≤) can be defined as:

$$(x \leq y) =_{def} (x < y) \vee (x = y) \quad \textbf{(4)}$$

[1] The Unified Modeling Language (UML) is a *de facto* standard for software and data engineering (http://www.uml.org/). A modeling profile is a built in mechanism in UML to create specialized versions of the language.

The axioms (1-3) define the minimal (partial ordering) constrains that every relation must fulfill to be considered a partOf relation. Although necessary, these constraints are not sufficient, i.e., it is not the case any partial ordering qualifies as a parthood relation. Some authors [6], require an extra axiom termed the *weak supplementation principle* (5) as constitutive of the meaning of part and, hence, consider (1-3) plus (5) (the so-called *Minimal Mereology* (MM)) as the minimal constraints that a mereological theory should incorporate.

$$\forall x,y\ (y < x) \rightarrow \exists z\ (z < x) \wedge \neg overlap(z,y) \qquad (5)$$

An extension to MM has then been created by strengthening the supplementation principle represented by (5). In this system, (5) is thus replaced by the following stronger supplementation axiom:

$$\forall x,y\ \neg(y \leq x) \rightarrow \exists z\ (z \leq y) \wedge \neg overlap(z,x) \qquad (6)$$

Formula (6) is named the *strong supplementation principle*, and the theory that incorporates (1-3), (5) and (6) is named *Extensional Mereology (EM)*. A known consequence of the introduction of axiom (6) is that in EM, we have that two objects are identical iff they have the same (proper) parts, a mereological counterpart of the *extensionality principle* (of identity) in set theory.

A second way that MM has been extended is with the aim to provide a number of closure operations to the mereological domain. As discussed, for example, in [6,7], theories named *CMM (Closure Minimal Mereology)* and *CEM (Closure Extensional Mereology)* can be obtained by extending MM and EM with the operations of *Sum*, *Product*, *Difference* and *Complement*. In particular, with an operation of sum (also termed *mereological fusion*), one can create an entity which is the so-called *mereological sum* of a number of individuals.

1.2. Problems with Mereology as a Theory of Conceptual Parts

Mereology has shown itself useful for many purposes in mathematics and philosophy [6,7]. Moreover, it provides a sound formal basis for the analysis and representation of the relations between parts and wholes regardless of their specific nature. However, as pointed out by [8,9] (among other authors), it contains many problems that make it hard to directly apply it as a theory of conceptual parts. As it shall become clear in the discussion that follows, on one hand the theory is too strong, postulating constraints that cannot be accepted to hold generally for part-whole relations on the conceptual level. On the other hand, it is too weak to characterize the distinctions that mark the different types of conceptual part-whole relations.

A problem with ground mereology is the postulation of unrestricted transitivity of parthood. As discussed in depth in the literature [1,10], there many cases in which transitivity fails. In general, in conceptual modeling, part-whole relations have been established as non-transitive, i.e., transitive in certain cases and intransitive in others.

The problem with extensional mereologies from a conceptual point of view arises from the introduction of the strong supplementation principle and, consequently, of formula (6) which states that objects are completely defined by their parts. If an entity is identical to the mereological sum of its parts, thus, changing any of its parts changes the identity of that entity. Ergo, an entity cannot exist without each of its parts, which is the same as saying that all its parts are *essential parts*. Essential parthood can be defined as a case of *specific constant dependence* (*existential dependence*) between individuals, i.e., x is an essential part of y iff y cannot possibly exist without having that specific individual x as part [6]. As discussed in depth in [2], essential parthood plays a fundamental role in conceptual modeling. However, while some parts of objects represented in conceptual models are essential, not all of them are essential. The failure to acknowledge that can be generalized as the failure of classical mereological theories to take into account the different roles that parts play within the whole. As discussed in [4,11], a conceptual theory of parthood should also countenance a *theory of wholes*, in which the relations that tie the parts of a whole together are also considered.

From a conceptual point of view, the problem with the theory of General (Classical) Extensional Mereology is related to the existence of a sum (or fusion) for any arbitrary non-empty (but non-necessarily finite) set of entities. Just as in set theory one can create a set containing arbitrary members, in GEM one can create a new object by summing up individuals that can even belong to different ontological meta-categories. For example, in GEM, the individual Θ created by the sum of Noam Chomsky's left foot, the first act of Puccini's Turandot and the number 3, is an entity considered as legitimate as any other. As argued by [9], humans only accept the summation of entities if the resulting mereological sum plays some role in their conceptual schemes. To use an example cited there: the sum of a frame, a piece of electrical equipment and a bulb constitutes an integral whole that is considered meaningful to our conceptual classification system. For this reason, this sum deserves a specific concept in cognition and name in human language. The same does not hold for the sum of bulb and the lamp's base. Once more, we advocate that a theory of conceptual parthood must also comprise a theory of wholes.

According to Simons [6], the difference between purely formal ontological sums and, what he terms, *integral wholes* is an ontological one, which can be understood by comparing their existence conditions. For sums, these conditions are minimal: the sum exists just when the constituent parts exist. By contrast, for an integral whole (composed of the same parts of the corresponding sum) to exist, a further *unifying condition* among the constituent parts must be fulfilled. A unifying condition or relation can be used to define a closure system in the following manner. A set B is a closure system under the relation R, or simply, R-closure system iff

$$\mathbf{cs}\ \langle R \rangle\ B =_{def} (\mathbf{cl}\ \langle R \rangle\ B) \wedge (\mathbf{con}\ \langle R \rangle\ B) \qquad (7)$$

where **(cl ⟨R⟩ B)** means that the set B is closed under R (R-Closed) and **(con ⟨R⟩ B)** means that the set B is connected under R (R-Connected). R-Closed and R-Connected are then defined as:

$$\mathbf{cl}\ \langle R \rangle\ B =_{def} \forall x\ (x \in B) \rightarrow ((\forall y\ R(x,y) \vee R(y,x) \rightarrow (y \in B)) \qquad (8)$$
$$\mathbf{con}\ \langle R \rangle\ B =_{def} \forall x\ (x \in B) \rightarrow (\forall y\ (y \in B) \rightarrow (R(x,y) \vee R(y,x)) \qquad (9)$$

An integral whole is then defined as an object whose parts form a closure system induced by what Simons terms a *unifying* (or *characterizing*) *relation* R.

Besides modal meta-properties such as generic or specific dependence of parts, conceptual modeling theory of parthood must also recognize additional modes in which something can be a part of a whole. One of these notions of great importance is the so-called (*non-)shareability* (or *exclusiveness*) of parts [4]. This distinction is reflected, for instance, in UML as the distinction between the aggregation relation (represented as a hollow diamond) and a composition relation (represented as a black diamond). In a nutshell, an individual x of type A is said to be an non-shareable (proper) part of another individual y of type B (symbolized here as $<_{NS}(x,A,y,B)$) iff y is the only B that has x as part:

$$<_{NS}(x,A,y,B) =_{def} \mathbf{instanceOf}(x,A) \land \mathbf{instanceOf}(y,B) \land (x<y) \land (\forall z\ \mathbf{instanceOf}(z,B) \land (x < z) \rightarrow (y = z)) \quad (10)$$

Finally, in a seminal article entitled *"A taxonomy of part-whole relations"*, Winston & Chaffin & Herrmann [12] (henceforth WCH), propose an account of the notion of parthood by elaborating on different ways that parts can related to a whole. This study led to a refinement on the formal relation of parthood by distinguishing the six types of meronymic relations. This taxonomy has been later refined by [8] demonstrating that the six linguistically-motivated types of part-whole relation proposed give rise to only three distinct ontological types, namely: (a) *subquantity-quantity* (e.g., alcohol-wine) – modeling parts of an amount of matter; (b) *member-collective* (e.g., a specific tree – the black forest) – modeling a collective entity in which all parts play an equal role w.r.t. the whole; (c) *component – functional complex* (e.g., heart-circulatory system, engine – car) - modeling an entity in which all parts play a different role w.r.t. the whole, thus, contributing to the functionality of the latter.

2. Homeomerosity and Identity related to Quantities

In [13], van Leeuwen shows an important syntactical difference in natural languages that reflects a semantical and ontological one, namely, the difference between common nouns (CNs) on one side and arbitrary general terms (adjectives, verbs, mass nouns, etc…) on the other. CNs have the singular feature that they can combine with determiners and serve as argument for predication in sentences such as:

(i) (exactly) five mice were in the kitchen last night;
(ii) the mouse which has eaten the cheese, has been in turn eaten by the cat.

In other words, if we have the patterns *(exactly) five X…* and *the Y which is Z…*, only the substitution of X,Y,Z by CNs will produce sentences which are grammatical. To see that, we can try the substitution by the adjective *Red* in the sentence (*i): (exactly) five red were in the kitchen last night*. A request to 'count the red in this room' cannot receive a definite answer: Should a red shirt be counted as one or should the shirt, the two sleeves, and two pockets be counted separately so that we have five reds? The problem in this case is not that one would not know how to finish the counting but that one would not know how to start since arbitrarily many parts of a red thing are still red.

The distinction between the grammatical categories of CNs and arbitrary general terms can be explained in terms of the ontological categories of *Sortal* and *Characterizing universals* [14] (also termed *mixin types* in the object-orientation literature [3]), which are roughly their ontological counterparts. Whilst the latter supply only a *principle of application* for the individuals they collect, the former supply both a principle of application and a *principle of identity*. A principle of application is that in accordance with which we judge whether a general term applies to a particular (e.g. whether something is a Person, a Dog, a Chair or a Student). A principle of identity supports the judgment whether two particulars are the same, i.e., in which circumstances the identity relation holds. By supplying a principle of identity and individuation, a sortal can also supply a principle of counting: counting depends on identity since to count correctly one cannot count *the same* individual twice.

The statement that we can only make identity and quantification statements in relation to a Sortal amounts to one of the best-supported theories in the philosophy of language, namely, that the identity of an individual can only be traced in connection with a Sortal Universal, which provides a *principle of individuation* and *identity* to the particulars it collects [13]. The Sortal supplying these principles is named a Substance Sortal [13] or a *Kind* [3]. In [3], the authors advocate an equivalent stance for a theory of conceptual modeling by defending that: (i) among the conceptual modeling counterparts of general terms (classifiers), only constructs that represent substance sortals (kinds) can provide a principle of identity and individuation for its Instances; (ii) every individual in a conceptual model must be an instance of a sortal.

One difference between subquantity-quantity and the other two types of parthood is that the relata of this relation always belong to the category of *amounts of matter* (masses, quantities), while in the component-functional complex and member-collective they are *Substantials* (or *Objects*) [4]. Quantities (such as water, sand, sugar, martini, wine, etc.) lack both *individuation* and *counting principles*. For this reason, the general terms which are linguistically represented by mass nouns (the linguistic counterpart of amounts of matter) cannot be used to substitute X, Y and Z in the aforementioned sentence patterns. A substitution for, for example, water in sentence (i) is not viable, since arbitrarily many parts of water are still water. Likewise, a success in the substitution by water in (ii) depends on the possibility of determining the referent and judge identity statements of individual quantities of water. What exactly should be that referent?

Before answering this question we should call attention to the notion of *homeomerosity*, which is used both by the WCH taxonomy and by [8] to distinguish subquantity-quantity relations from the other two types aforementioned. Traditionally, homeomerosity means that an individual only has parts which are of the same kind [15]. This is clearly not the case for all amounts of matter, as the Gin-Martini case demonstrates. However, one can still say that every subquantity of Martini is again Martini and that although Martini is composed of Gin, Gin is itself “homeomerous” in this more liberal sense. This line of reasoning seems to suggest that homeomerosity is equated with infinite decomposability, i.e., for every subquantity of Martini there is always a subquantity of Martini, and the same holds for quantities of Gin. Some authors (e.g., [15]), nonetheless, admit the existence of quantities of type K having K-atoms, i.e., individuals of type K that have no parts of the same type K. Examples include concrete mass terms such as ‘furniture’, ‘cutlery’ or ‘crowd’. These allegedly exemplars of quantities are definitely not homeomerous, not even in the more liberal sense. For example, there are parts of a crowd, namely individual persons, which are not a crowd themselves and which are not homeomerous in any meaningful sense. What can be said in this case is that these aggregates have a uniform structure and, in parity with [8], we consider them as examples of *member-collective* parthood instead.

One could also consider homeomerosity to simply mean that an aggregate can merely have some parts of the same kind while having other parts of other kinds. However, if this were to be the case one could not use it as a meta-property to differentiate subquantity-quantity from member-collective. Notice that examples of "homeomerous" parts in this sense can be easily found for member-collective: a crowd can be part of a larger crowd; a forest can be part of larger forest. Since WCH do not consider member-collective to be homeomerous, they would have to agree that quantities should be considered necessarily infinitely divisible in subquantities of the same kind.

Now, an important question that comes to the mind is how we should represent in conceptual models the types whose instances are quantities in the sense just mentioned? As we have discussed in depth in [3,13], in order to be able to make viable references to general terms which are not count nouns (mass terms, adjectives, verbs) they first must be nominalized. A nominalization of a mass noun, verb or an adjective promotes the shift to the category of count nouns (e.g., the fall of Jack, a lump of clay), hence, allowing for the representation of the corresponding (nominalized) sortal type. An important question that then arises is: what is the best nominalization of mass terms so that they can be satisfactorily represented in conceptual models as sortal types? In the next section, we investigate three candidate nominalization/representation alternatives.

3. On the Representation of Quantities in Conceptual Modeling

3.1. Quantities as Mereological Sums

In order to investigate the representation alternatives let us take the example of a portion of wine which could be differentiated from other portions of wine by the year and source vineyard. What is the meaning (and implied principle of individuation) of a universal whose instances are portions of wine?

A first possibility is to consider the referent of the expression "the portion of wine" as a mereological fusion of all subportions of wine that constitutes it. This approach is standard in philosophy and [6] suggests that quantities are probably the best case of application of the Classical Extensional Mereologies (CEM), since practically all objections raised against the CEM for the purpose of conceptual modeling can be safely lifted in the case of quantities and their parts. Nonetheless, and still from a philosophical point of view, the first problem with this conception of quantities is whether it is at all possible to have a principle of identity for portions of wine in this sense [15,16]. A mereological principle of identity in this case prescribes that portion of wine A is equal to portion of wine B iff they have the same parts. However, since the parts of A and B are also portions of wine, to decide if A and B have the SAME parts one has to decide about the identity of the parts, and the parts of the parts, leading to an infinite regress, since, by assumption, quantities are infinitely divisible. One could derive some synchronic information about identity by saying that two quantities are different if they do not occupy the same region of space. However, this cannot be used as a diachronic principle of identity. Alternatively, one could say that a quantity A in t_1 is not the same quantity as B in t_2 if they have different properties such as volume or weight (still assuming the mereological principle). Nonetheless, the fact that B in t_2 has the same volume or weight as A in t_1 can only account for the sameness of the quantities in a very loose sense, meaning the same measure. In other words, in this case, the relation between A and B is one of equivalence, not one of *numerical identity*.

In conceptual modeling, there are a number of situations in which dealing only with qualitative identity of masses does not suffice. For instance, one might be interested in tracking the persistence of a quantity of a certain liquid which has been poisoned, or, in a chemical experiment, it could be important to track the change of properties in the very SAME persisting quantity. For this reason, contra [8], who proposes that "quantities are arbitrary pieces of the whole as long as they are properly characterized by the quantitative measure", we advocate that a treatment of masses in conceptual modeling must deal explicitly with the case of numerical identity.

However, there is still a bigger problem with this idea from a conceptual modeling stance. Figure 1 depicts the representation of a portion of Wine universal in the sense just mentioned. In this specification, the idea is to represent a certain portion of wine as the mereological sum of all subportions of wine belonging to a certain vintage. As it can be noticed, since every portion of wine is composed of subportions of wine, the cardinality of the part-whole relation cannot be specified in a finite manner. The same holds for every cardinality constraint for associations involving portions of wine. As discussed, for instance in [17], *finite satisfiability* is a fundamental requirement for conceptual models which are intended to be used in areas such as Databases and Software Engineering.

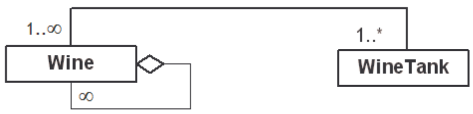


Figure 1. Representing Quantities as Mereological Sums.

Furthermore, "homeomerous" entities represented in this manner can induce to representation errors in the presence of other shareability constraints. For example, figure 2 presents an exact copy of a UML class diagram from [18][2] that symbolizes a Fractal (perhaps the prototypical example of homeomerous form). The intention of the authors seems to be to represent that a fractal, i.e., the rendering of one iteration step of an IFS (Iterated Function Series), is part of only one instance of the infinite recursion of this function. In other words, a part of an instantiation of a *Julia Series* is not a part of another instantiation of the same fractal form, or part of an instantiation of the *Mandelbrot Series*. However, that is not what is represented in the model. The model states that every instance of a fractal (i.e., every iteration of an IFS) is part of only one other fractal. This is clearly mistaken: the n^{th} iteration of an IFS is part of all previous iterations of the same fractal. This problem is far from being specific to Fractals. In fact, for all homeomerous entities, with the exception of the maximal sum of subquantities, all other parts of quantity are necessarily part of innumerable other quantities of the same kind.

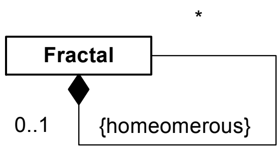


Figure 2. Mistaken Representation of homeomerous parts with non-sheareability (from [19]).

[2] The tagged value {homeomerous} in the picture is a proposal of [18] and it is present in the original article.

3.2. Quantities as the Identificationally Dependent on Objects

For the purpose of conceptual modeling, there is still one further philosophical argument invalidating the modeling alternative discussed in the previous section. According to [19], masses are *identificationally dependent* of substantials that are instances of sortal universals. As he put it: "The formal concepts 'amount', 'part' and 'stuff', like 'mass' and 'matter', are formed 'on the back of' [19, p. 191] the formal concept '(material) thing' or '(material) object'. There is no mass, except the mass of a certain object. There is no stuff except the stuff a certain thing consists of". This is to say that individuating a quantity depends on the definite descriptions used for referring, which succeeds only when the referent can be individuated by an object type, i.e., it is the water in the bathtub, the clay that constitutes the statues, a cube of sugar, that can be referred to, not just some water, some clay, or some sugar. Moreover, quantities have no criteria for when they constitute a whole of some sort (unity criteria), except in cases in which we derive those criteria from objects that are only identifiable via sortals. This view is also supported by Quine in [20], who proposes that every occurrence of a quantity expression having the form "The K", "The same K" (where K is a concrete quantity type) is really a masked reference to a *portion* of K to which an ordinary sortal type applies. Alternatively formulating, as Quine puts it: "[in these situations, always] some special individuation standard is understood from the circumstances". This perspective gives rise to a second option of representation for the wine/wine tank example, as depicted in figure 3.

Figure 3. Representing Quantities as *Identificationally Dependent* on Objects.

There are a number of observations that can be made about figure 3. In this second option for nominalization of quantities, Wine means the maximal content of a Wine Tank. Likewise, the referent of a portion of clay means whatever quantity of clay constitutes a given statue, which is in turn individuated by the principle of identity supplied by the sortal Statue. In this representation, there is no longer a problem for the specification of cardinality constraints between portion of wine and wine tank: every wine tank has as its content one single definite portion of wine. Additionally, since wine portion means the maximal content of a wine tank, it is not the case that this concept is homeomerous, i.e., there is no part of a portion of wine which is itself a portion of wine (otherwise, it would not be the maximal content). Thus, there is no problem with infinite cardinalities, infinite divisibility and, hence, the model can in principle be finitely satisfiable. Moreover, portion of Wine becomes a type supplying genuine principles of individuation, counting and identity: it is always determinate if two portions of wine are identical and it is always determinate how many portions of wine there are.

Now, what is the nature of the relation R between a quantity of Wine and WineTank in figure 3? It is more than a relationship of spatial containment. Notice that if the identity of the quantity is defined by its container than relation R becomes one of existential dependency. Take the statue/lump of clay example. If A is the same lump of clay as long as it constitutes the same statue B, A would have a complete life-time dependency to B. For instance: (a) if a piece of B is removed, B is still the same statue and so is A still the same lump of clay, since it still constitutes the same statue; (b) If the form of B is altered, B ceases to exist and so does A, since it no longer constitutes the same statue. In summary, in this second alternative nominalization, we have to arrive at the counter-intuitive conclusion that a quantity in this sense has properties which are more akin to the notion of *moment* (*trope*) than that of substantial (object) [4].

3.3. Quantities as Maximally Self-Connected Objects

Although this second alternative contains important advantages over the first one from a conceptual modeling point of view, it leads to a problematic consequence. The problem is implied exactly by the *rigid specific dependence* relation between a quantity in this sense and its container. As put by [21], a sentence such as the "same K" (where K is a quantity universal) should be understood in a such way that **x is the same K as y iff x is some K, y is some K, and (x = y),** or, as discussed in [3,13], in statements of identity, the relata must instantiate the same kind, i.e., the same rigid sortal supplying their principle of identity. In parity with [15, 21], we consider as meaningful a sentence such as "the sugar that was in that cube is the same sugar as the one in this lump". However, if this is the case, which kind of individuation and identity principle should be applied to x and y, that of cubes or of lumps?

A third nominalization alternative that solves this problem is presented as follows. This last option relies on the notion of *piece* discussed by Lowe in [16]. According to Lowe, a piece of a quantity K is a maximally self-connected object constituted by portions of K (portions in the first sense discussed in 3.1). Following Lowe, by maximal self-connected portion of K we mean the following: (i) *connected*, in the sense that every part of it is spatially connected to every other part of it by a series of spatially contiguous parts; (ii) *maximal*, in the sense that it is not a proper part of any larger connected part of stuff of the same kind K. One should notice that this relation of (spatially contiguous) connection is transitive: if x is connected to y by a series of spatially contiguous parts, and if y is connected to z in the same sense, we have that x is spatially-contiguously connected to z.

Like in the second nominalization alternative, a quantity of K in this sense is an instance of a type supplying definite individuation, identity and counting principles. Moreover, it is not homeomerous, however it can still be composed of other quantities K' in the same sense of quantity (see figure 4). Moreover, in this case, it is still convenient to consider all parts of quantity as essential: a specific quantity of Wine is composed by that specific quantity of Alcohol. Finally, this representation alternative also does not contain the infinite regress problems mentioned for the first case.

Now, differently from the second alternative, the dependence relation between a quantity and its container is a generic not a specific one. For this reason we can state that for the same maximally self-connected quantity of wine, there can be several "container phases". This idea is represented in the (incomplete) model of figure 4. A vintage is an (substantial) object constituted by (possibly many) quantities of wine. It is, however, not a quantity since it can be scattered over many quantities. Moreover, it is not necessary for its constituent quantities to be essential: even if the quantity of wine now stored in a certain tank is destroyed, we still have numerically the same vintage. We, therefore, propose the use of this third alternative for the nominalization of quantities and their representation in conceptual models. From now on, we shall use the term *quantity of matter K* or *objectified portion of matter K* to refer to a *piece of K* in the Lowe's sense aforementioned, and use the stereotype «quantity» to symbolize a (sortal) type whose instances are quantities in this sense.

In the first representation alternative, quantities have very minimal existence conditions and can hardly be said to constitute integral wholes. In the second representation alternative, a genuine characterizing or unifying relation can be defined creating an integral whole. However, this relation is an external one, creating a specific dependence to a container object. Moreover, in this second alternative, a quantity resembles more a moment than an object, becoming identificationally and existentially dependent on its container. In the third representation choice, quantities are object-like integral wholes, unified by an intrinsic and again genuine unifying relation of spatially contiguous maximal self-connectedness.

Finally, we can summarize many points of the argument carried out in this section by using an example proposed by [21]. If a sentence such as "Heraclitus bathed in some water yesterday and bathed in the same water today" is true then for some suitable substituends of x and y we have that: (a) x is a quantity of water and Heraclitus bathed in x yesterday; (b) y is a quantity of water and Heraclitus bathed in y today; (c) x = y. However, (c) when interpreted as (d) "the water Heraclitus bathed in yesterday = the water Heraclitus bathed in today" requires that: (e) there is exactly one x such that x is a quantity of water and Heraclitus bathed in x both yesterday and today. But, if quantity is interpreted in the first sense discussed in 3.1, there are not one but infinitely many particulars that would satisfy (a), (b) and (d) without satisfying (c). The same does not hold for the second and third senses.

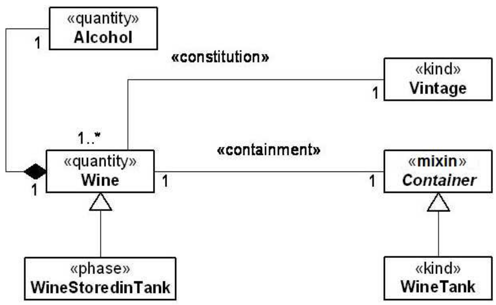


Figure 4. Representing quantities as maximally-self-connected portions

4. Representing parts of a Quantity: the subQuantityOf relation

We can now define the *subQuantityOf* as a part-whole relation holding between *quantities* in the technical sense explained in the previous section. As depicted in figure 5 below, we decorate here the standard UML symbol for composition with a Q to represent this relation.

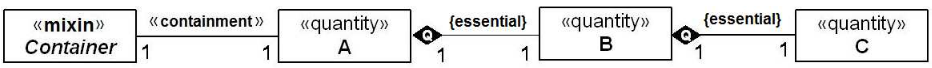


Figure 5. Part-Whole relations among Quantities.

Let us take the quantities A, B and C represented in this figure. We can show that for any A, B, C, the part-whole relation (C < A) holds as a result from the transitivity (C <

B) and (B < A). The argumentation can be developed as follows: if A is a quantity then it is a maximal portion of matter unified by the characterizing relation of self-connectedness. That is, any part of A is connected to any other part of A. If B is part of A then B is connected to all parts of A. Likewise, if C is part of B then C is connected to all parts of B. Since (as previously discussed) this notion of spatially contiguous connection is transitive, then we have that C is connected to all parts of A. Thus, since A is unified under self-connection, C must be part of A (otherwise the composition of A would not be a closure system, see definition in section 2). Therefore, we conclude that for the case of quantities, transitivity always holds. Another way to examine this situation is by inspecting A at an arbitrary time instant t. We can say that all parts of A are the quantities that are contained in a certain region of space R (i.e. a topoid[3]). Since A is an objectified matter, than the topoid R occupied by A must be self-connected. Therefore, if B is part of A then B must occupy a sub-region R', which is part of R. Likewise, if C is part of B, it occupies a region R'', part of R'. Since spatial part-whole relations are always transitive [22], we have that R'' is part of R, and if C occupies R'', then it is contained in R. Ergo, by definition, C is a part of A.

As discussed in section 3.3, a type stereotyped as « quantity » in this work stands for a maximally-connected-amount-of-matter. Since a quantity is maximal, it cannot have as a part a quantity of the same kind. For the same reason, a *subQuantityOf* relation is always non-shareable. For example, take a case in which this relation holds between a quantity of alcohol *x* and a quantity of wine *y*. Since y is self-connected it occupies a self-connected portion of space. The same holds for x. In addition, the topoid occupied by x must be a (improper) part of the topoid occupied by y. Now suppose that there is a portion of wine z (different from y) such that x is a subQuantityOf z. A consequence of this is that z and x overlap, and since they are both self-connected, we can define a portion of wine w which is itself self-connected. In this case, both z and x are part of w and therefore, they are not *maximally-self-connected-portions*. This contradicts the premises that x and z were quantities. Hence, we can conclude that the *subQuantityOf* relation is always non-sharable. Furthermore, since every part of a quantity is itself a quantity, subQuantityOf must also have a cardinality constraint of *one and exactly one* in the subquantity side. Take once more the alcohol-wine example above. Since alcohol is a quantity (and, hence, maximal), there is exactly one quantity of alcohol which is part of a specific quantity of wine.

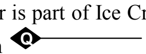
Also as discussed in section 3.3, quantities are mereologically invariant, i.e., the change of any of its parts changes the identity of the whole. In other words, all *subQuantityOf* relations are essential parthood relations (see relations tagged with {essential} in figure 5). Therefore, since quantities are extensional entities, the *weak supplementation axiom* (4) defined to hold for all part-whole relations can be replaced by the adoption of the *strong supplementation axiom* for the case of this relation.

The axiomatization of the *subQuantityOf* relation thus includes the basic axioms of any mereological theory, namely, irreflexivity, anti-symmetry and transitivity of the proper-part relation. But also the *strong supplementation axiom* and the *extensionality principle*. Moreover, it includes the exclusive parthood and the essential parthood axioms. In other words, the axiomatization of this relation is the one of *Extensional Mereology (EM)*, with non-shareable and essential parts. That means that two quantities are the same iff they have the same parts, and no two instance of the same quantity kind overlap.

[3] A Topoid is a region of space with a certain mereotopological structure.

We summarize the results of these sections in a proposal that can be incorporated in a UML profile for representing subQuantityOf relations (table 1). Since a profile is constituted by syntactical constraints and, since UML conceptual models are always defined at the type level, the meta-properties of irreflexivity, anti-symmetry and transitivity (at instance level) cannot be captured by profile constraints. We have included a constraint to guarantee weak supplementation for these relations taking in consideration the type-level nature of the diagrams, i.e., taking into consideration the minimum cardinality constraints of all subQuantityOf relations connected to the same type representing a whole. Finally, one should notice that the diagrams of figure 4 and 5 do not obey this constraint and should be interpreted as (incomplete) model fragments.

Table 1. Metamodel Constraints for a UML profile for modeling Quantities and the subQuantityOf relation

Metaclass	**Description and Concrete Syntax**
«quantity» **A**	A «quantity» represents a sortal whose instances are *quantities*. Examples are those stuff universals that are typically referred in natural language by mass general terms (e.g., Gold, Water, Sand, Clay).
subQuantityOf	subQuantityOf is a proper parthood relation between two quantities. Examples include: (a) alcohol is part of Wine; (b) Plasma is part of Blood; (c) Sugar is part of Ice Cream; (d) Milk is part of Cappucino. We propose the icon  to represent this relation.

Meta-Properties of subQuantityOf

Irreflexivity, Asymmetry, Transitivity and Strong Supplementation (Extensional Mereology).

Constraints for subQuantityOf

1. The classes connected to both association ends of this relation must represent universals whose instances are *quantities*, i.e., they must be either stereotyped as «quantity» or be a *subtype* of a class stereotyped as «quantity»;
2. This relation is always non-shareable (always represented with the black diamond notation in UML);
3. All entities stereotyped as «quantity» are extensional individuals and, thus, all parthood relations involving quantities are marked with the tagged value **{essential}** representing an essential parthood relation. As a consequence, the association end connected to the part must be immutable;
4. The maximum cardinality constraint in the association end connected to the part must be one;
5. Weak Supplementation: Let U be a universal whose instances are wholes and let $\{C_1 \ldots C_2\}$ be a set of universals related to U via aggregation relations. Let $lower_{C_i}$ be the value of the minimum cardinality constraint of the association end connected to C_i in the aggregation relation. Then, we have that

$$\left(\sum_{i=1}^{n} lower_{C_i}\right) \geq 2;$$

5. Final Considerations

The development of suitable foundational theories is an important step towards the definition of precise real-world semantics and sound methodological principles for conceptual modeling languages. In this paper, we conduct an ontological analysis to investigate the proper representation of types whose instances are quantities, as well as the representation of parthood relations involving quantities. As result, we were able to provide not only a sound ontological interpretation for the notion of quantity types, but also one that satisfies two fundamental modeling requirements: determinate numerical identity for their instances; finite satisfiability for conceptual models representing quantity types and subQuantityOf relations. In addition, the results advanced here contribute to the definition of concrete engineering tools for the practice of conceptual modeling. In particular, the metamodel extensions and associated constraints proposed can be implemented using available UML metamodeling tools in a straightforward manner (as demonstrated in [23]). This extended UML metamodel, in turn, can be directly employed by automated tools to support the development of conceptual models which are sensible to the ontological notions discussed here. Finally, the representation for the notion of quantity and subQuantityOf put forth here have been employed successfully in an industrial case study in the domain of petroleum and gas [24] to model notions such as specific quantities of *Reservoir Rock* and its subparts (specific subquantities of Water, Gas and Oil) as well as the participation of quantities in events such as Petroleum Production and Transportation.

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